General Relativity For Tellytubbys

Einstein Curvature

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Overview

This section gets to grips with *the* fundamental tensor of G.R. The fundamental postulate is that the stress-energy tensor is equal to a tensor that measures geodesic deviation. This postulate is motivated by the equivalence of gravitational and inertial mass, i.e. all objects fall with the same acceleration in the same gravitational field. This is normally expressed as mass determines the shape of your girlfriend's tits and arse, and then that shape tells one if one wants to move your hands in that general area. This seems so obvious, in hindsight, that it makes you wonder why it took Einstein, genius that he was, some bloody 10 years to figure it out.

The stress energy tensor satisfies statements of conservation of momentum and energy

$$\nabla \mathbf{T} = \mathrm{T}_{\alpha\beta;\beta} = 0$$

So, obviously we need a geodesic deviation tensor that also satisfies this equation. Riemann by itself is a bit big being a 4-rank tensor, but fortunately it has so many symmetries that a contracted version of Riemann does the job no sweat, with no loss of geodesic deviation information.

First, the result will be stated, then we'll do some more hand waving to derive the results.

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \lambda g_{\alpha\beta}$$

Where:

$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$ and $R = R^{\alpha}{}_{\alpha} = g^{\alpha\beta} R_{\alpha\beta}$

i.e.

$$G_{\alpha\beta;\beta} = (R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \lambda g_{\alpha\beta})_{;\beta} = 0$$

So that

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

where the 8, G, pi, are fudge factors to make the equation agree with Newton in the low field, low velocity limit. Note the two G's to confuse you. The one without the indexes is the Newton gravitational constant.

The G tensor is called the Einstein tensor, but he was not the first to derive it. The lambda term is the "greatest blunder of my life" term i.e. the cosmological constant, which was first absent then added then removed then added, and so it goes on and on…

Construction of The Einstein Tensor

From the Riemann section, we have the Bianchi Identity:

$$R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} = 0$$

Now define the Ricci 2nd Rank Tensor and Ricci Scalar, obtained by contracting the Riemann Tensor:

$$R_{bd} = R^c{}_{bcd}$$

$$R = R^d{}_d$$

Contracting the Riemann Bianchi Identity by setting a=c gets us:

$$R^c{}_{bcd;e} + R^c{}_{bec;d} + R^c{}_{bde;c} = 0$$

$$R_{bd;e} - R_{be;d} + R^c{}_{bde;c} = 0$$

Contracting on b and d

$$R^d{}_{d;e} - R^d{}_{e;d} + R^{cd}{}_{de;c} = 0$$

But by the antisymmetry of the 1st 2 indexes of Riemann

$$R^d{}_{d;e} - R^d{}_{e;d} - R^{dc}{}_{de;c} = 0$$

$$R^d{}_{d;e} - R^d{}_{e;d} - R^c{}_{e;c} = 0$$

But the last terms can be seen to be dummy indexes and so,

$$R^d{}_{d;e} - R^d{}_{e;d} - R^d{}_{e;d} = 0$$

$$R^d{}_{d;e} - 2R^d{}_{e;d} = 0$$

$$R_{;e} - 2R^d{}_{e;d} = 0$$

or

$$R^d{}_{e;d} - \frac{1}{2}R_{;e} = 0$$

$$R^d{}_{e;d} - \frac{1}{2}\delta_e{}^d R_{;d} = 0$$

$$R^d{}_{e;d} - \frac{1}{2}\delta_e{}^d R_{;d} = 0$$

$$(R^d{}_e - \frac{1}{2}\delta_e{}^d R)_{;d} = 0$$

$$g^{ae}(R^d{}_e - \frac{1}{2}\delta_e{}^d R)_{;d} = 0$$

$$(R^{da} - \frac{1}{2}g^{da}R)_{;d} = 0$$

hence defining the Einstein Tensor as:

$$G^{ab} = R^{ab} - \frac{1}{2}g^{ab}R + \lambda g^{ab}$$

one has

$$G^{ab}{}_{;a} = 0$$

Where the metric tensor has been included as the Einstein fudge factor, as its covariant derivative is identically zero.